Computational properties of term rewriting with replacement restrictions*

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Abstract

We give a general formulation of the notion of *replacement restriction*, a concept which induces a *restricted rewrite relation* on a term rewriting system. Being a very general concept, we introduce and motivate properties which can be used to characterize some important families of replacement restrictions. We show how to approximate the lattice of replacement restrictions by the finite lattice of *context-sensitive* replacement restrictions. This allows us to eventually lift existing results on computational properties (termination, completeness, ...) of context-sensitive rewriting to non-trivial approximations of different classes of restricted rewriting. We also give useful results on confluence for restricted rewriting as an easy generalization of well-known results for unrestricted term rewriting.

Keywords: functional programming, replacement restrictions, term rewriting.

1 Introduction

When computing with Term Rewriting Systems (TRSs) it is useful to *restrict* the allowed reductions in order to permit only reduction steps which are useful to normalize a term. Such a restriction is usually given as a reduction strategy. Even if we are sure that any computation terminates, we are still interested in restricting the allowed reductions, in order to produce more efficient, unwasteful computations. In practice, non-terminating systems arise when we manipulate infinite data structures and so on. An obvious way to (try to) improve the terminating behavior of the system is to disallow reductions at some specific positions of a term. Roughly speaking, given a term t, we allow for reductions at the occurrences given by $O^\gamma(t) \subseteq O(t)$, where γ denotes the concrete *replacement restriction*. Reductions at occurrences in $\widetilde{O^\gamma}(t) = O(t) \backslash O^\gamma(t)$ are forbidden. Regretably, there are two aspects of the operational behavior which are in competence: termination and completeness of evaluations. Completeness implies the ability to compute results

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belonging to a signifiant class of terms (head-normal forms, values, normal forms, ...). Restrictions in reductions improve termination but can also compromise completeness.

The study of termination and completeness of general classes of rewriting induced by generic replacement restrictions is rather involved. These subjects have been studied for a particular, fixed class we call context-sensitive rewriting [7, 8, 12]. This kind of restriction can be defined on a purely syntactic basis, namely, by lifting to occurrences of terms, the replacement restrictions on the arguments of the symbols in a signature Σ. The replacement restrictions on Σ are denoted by the so called *replacement map* μ, a function $\mu : \Sigma \to \mathcal{P}(\mathbb{N})$. Being a context-sensitive replacement restriction easily computable, it is worthy to analyze how to approximate arbitrary, more complex replacement restrictions by using (combinations of) context-sensitive restrictions. We define the best upper and lower context-sensitive approximations for a given replacement restriction. We also show that combinations of context-sensitive restrictions can give better approximations.

Many interesting results on restricted rewriting easily follow by simply adapting the corresponding results for unrestricted rewriting. For instance, in [7] we obtained a result on confluence of context-sensitive rewriting by introducing the replacement map μ in the Huet's proof of confluence of terminating TRSs having joinable critical pairs [3]. Here we obtain a similar result about confluence of generic restricted rewriting by generalizing the well known result on confluence of orthogonal TRSs. Unlike the method in [7], instead of considering a particular replacement restriction, we deduce which are the properties that are required to prove the result. In this way, we obtain the whole power of the proof method. This suggests the idea that concrete computational properties of generic restricted rewriting can easily be proved by generalizing the already established theory of rewriting: we try to formulate/generalize the proofs by using a set of basic properties to describe families of replacement restrictions.

This paper is organized as follows. In Section 2, we review the technical concepts and results used in the remainder of the paper. In Section 3, we introduce the notion of replacement restriction and motivate a set of basic properties that characterize it. In Section 4, we show how to use context-sensitive restrictions to approximate arbitrary restrictions. Section 5 analyzes computational properties of restricted rewriting, in particular confluence. Section 6 concludes.

2 Preliminaries

Let us first introduce the main notations used in the paper [2, 5]. V denotes a countable set of variables and Σ denotes a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each with a fixed arity given by a function $ar : \Sigma \to \mathbb{N}$. By $\mathcal{T}(\Sigma, V)$ we denote the set of terms. A k-tuple $t_1, \ldots, t_k$ of terms is denoted as $\tilde{t}$, where k will be clarified by the context. $Var(t)$ is the set of variable symbols of a term t.

Terms are viewed as labelled trees in the usual way. Occurrences $u, v, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. Occurrences are ordered by the standard prefix ordering: $u \leq v$ iff $\exists v'$ such that $v = u.v'$. $u \parallel v$ means $u \not\leq v$ and $v \not\leq u$. $O(t)$ denotes the set of occurrences of a term t. $O_\Sigma(t)$ are the occurrences of non-variable symbols in t. $t|_u$ is the subterm at occurrence u of t. $O_s(t)$ denotes the set of occurrences of s in t, i.e., $u \in O_s(t)$ iff $t|_u = s$. $t[s]_u$ is the term t with the subterm at the occurrence u replaced with s. We refer to any term C, which is the same as t everywhere except below u, i.e. there exists a term s such that $C[s]_u = t$, as the *context* within the replacement occurs. A context is a term C with a 'hole' at a

specific occurrence u.

A rewrite rule α is an ordered pair (l, r), written $\alpha : l \to r$ or just $l \to r$, with $l, r \in \mathcal{T}(\Sigma, V)$, $l \notin V$ and $Var(r) \subseteq Var(l)$. l is the left-hand side (*lhs*) of the rule and r is the right-hand side (*rhs*). A TRS is a pair $\mathcal{R} = (\Sigma, R)$ where R is a set of rewrite rules. An instance $\sigma(l)$ of a *lhs* l of a rule $l \to r \in R$ is a redex. $O_{\mathcal{R}}(t) = \{u \in O(t) \mid \exists l \to r \in R : t|_u = \sigma(l)\}$ denotes the set of redex occurrences in a term t, . t rewrites to s (at the occurrence u), written $t \to_{\mathcal{R}} s$, if $t|_u = \sigma(l)$ and $s = t[\sigma(r)]_u$, for some rule $\alpha : l \to r \in R$, $u \in O(t)$ and substitution σ. This can eventually be detailed by writing $t \overset{[u,\alpha]}{\to}_{\mathcal{R}} s$ or just $t \overset{u}{\to}_{\mathcal{R}} s$.

$\mathbb{N}_k^+$ is an initial segment $\{1, 2, \ldots k\}$ of the set of positive natural numbers $\mathbb{N}^+$, where $\mathbb{N}_0^+ = \emptyset$. $\mathcal{P}(\mathbb{N})$ is the powerset of natural numbers.

3 Replacement restrictions in rewriting

A replacement restriction γ is intended to limit the set of occurrences which can eventually be considered for reductions. We let $\gamma : \mathcal{T}(\Sigma, V) \to \mathcal{P}(\mathbb{N}^*)$ denote a function whose application to a term is denoted as $O^\gamma(t)$ (by notational uniformity) and is constrained by $O^\gamma(t) \subseteq O(t)$ for all $t \in \mathcal{T}(\Sigma, V)$. $O^\gamma(t)$ is the set of *γ-replacing* occurrences. $\widetilde{O^\gamma}(t) = O(t) \backslash O^\gamma(t)$ is the set of non γ-replacing occurrences. The set of replacement restrictions for a signature Σ is $\Gamma(\Sigma) = \{\gamma \in \mathcal{T}(\Sigma, V) \to \mathcal{P}(\mathbb{N}^*) \mid \forall t. O^\gamma(t) \subseteq O(t)\}$, or just Γ.

We lift the ordering $\subseteq$ on $\mathcal{P}(\mathbb{N}^*)$ to an ordering $\sqsubseteq$ on these functions: $\gamma \sqsubseteq \gamma' \Leftrightarrow O^\gamma(t) \subseteq O^{\gamma'}(t)$ for all term t. $(\Gamma, \sqsubseteq, \gamma_\top, \gamma_\bot, \sqcup, \sqcap)$ (or just Γ) is a distributive complete lattice [1]: for all t, $O^{\gamma_\top}(t) = O(t)$, $O^{\gamma_\bot}(t) = \emptyset$, $O^{\gamma \sqcup \gamma'}(t) = O^\gamma(t) \cup O^{\gamma'}(t)$ and $O^{\gamma \sqcap \gamma'}(t) = O^\gamma(t) \cap O^{\gamma'}(t)$. $\sqcup$ and $\sqcap$ are idempotent and distributive: $\gamma \sqcup \gamma = \gamma$, $\gamma \sqcap \gamma = \gamma$, $\gamma \sqcup (\gamma' \sqcap \gamma'') = (\gamma \sqcup \gamma') \sqcap (\gamma \sqcup \gamma'')$ and $\gamma \sqcap (\gamma' \sqcup \gamma'') = (\gamma \sqcap \gamma') \sqcup (\gamma \sqcap \gamma'')$.

$O_s^\gamma(t) = O^\gamma(t) \cap O_s(t)$ is the set of replacing occurrences of subterm s in t. $O_\Sigma^\gamma(t) = O_\Sigma(t) \cap O^\gamma(t)$ are the replacing occurrences of non-variable subterms. $Var^\gamma(t) = \{x \in Var(t) \mid O_x^\gamma(t) \neq \emptyset\}$ is the set of replacing variables in t.

Given a TRS $\mathcal{R} = (\Sigma, R)$, and $\gamma \in \Gamma$, $\hookrightarrow_{\mathcal{R}(\gamma)}$ (or just $\hookrightarrow_\gamma$ or $\hookrightarrow$ if no confusion arises) is the (one-step) *restricted rewrite relation*: t γ-rewrites to s, written $t \hookrightarrow_{\mathcal{R}(\gamma)} s$, if $t \overset{u}{\to}_{\mathcal{R}} s$, and $u \in O^\gamma(t)$. We have the following obvious property.

Proposition 3.1 (Monotonicity of $\hookrightarrow$ with respect to $\sqsubseteq$) *Let $\mathcal{R} = (\Sigma, R)$ be a TRS and $\gamma, \gamma' \in \Gamma$. Then $\gamma \sqsubseteq \gamma' \Rightarrow \hookrightarrow_{\mathcal{R}(\gamma)} \subseteq \hookrightarrow_{\mathcal{R}(\gamma')}$.*

Whenever a replacement restriction γ is considered, the concept of *redex* is not as 'operational' as in unrestricted rewriting. The most important property of a redex is that, in any term, any occurrence of this redex can be rewritten. If we consider a replacement restriction, this changes: we need to check whether the considered occurrence satisfies the replacement restriction. Thus, we call γ-redex to an *effectively* replacing redex in a term. The set of γ-redexes of a term is $O_{\mathcal{R}}^\gamma(t) = O_{\mathcal{R}}(t) \cap O^\gamma(t)$.

A property PROP of replacement restrictions is identified with a subset Γ^{PROP} of Γ, i.e., $\Gamma^{\mathsf{PROP}} \subseteq \Gamma$. We say that γ has the property PROP if $\gamma \in \Gamma^{\mathsf{PROP}}$. Given properties PROP1, ..., PROPn, the set $\Gamma^{\mathsf{PROP1}+\cdots+\mathsf{PROPn}} = \Gamma^{\mathsf{PROP1}} \cap \cdots \cap \Gamma^{\mathsf{PROPn}}$ denotes the set of restrictions which simultaneously satisfies PROP1, ..., PROPn.

3.1 Restricted rewriting and needed rewriting

Having no way to decide whether a single redex is needed to produce a normal form, some TRSs are constrained to be interpreted *in parallel* [9]. The *parallel or*,

$\mathtt{or}(\mathtt{true},\mathtt{x}) \to \mathtt{true}$ $\quad$ $\mathtt{or}(\mathtt{false},\mathtt{false}) \to \mathtt{false}$

$\mathtt{or}(\mathtt{x},\mathtt{true}) \to \mathtt{true}$

seems to require parallel evaluation: given a term $\mathtt{or}(cond, cond')$, assume that one argument evaluates to `true` and the evaluation of the other one runs into an infinite derivation. This prevents the value `true` from being produced. Since do not know which is the 'good' argument, we cannot safely evaluate sequentially.

The main result of Huet and Levy's [4, 6] is to give a formal basis for the definition of efficient *sequential* strategies, i.e., reduction sequences s.t. only one redex is reduced in each step. The basic idea is to represent *unknown* parts of a term t by using a new symbol Ω. Terms in $\mathcal{T}(\Sigma \cup \{\Omega\}, V)$ are said Ω-terms. Denote as $O_\Omega(t)$ the set of occurrences of Ω in t: $O_\Omega(t) = \{u \in O(t) \mid t|_u = \Omega\}$. To discuss about unknown portions of expressions, an ordering $\leq$ on Ω-terms is given: $\Omega \leq t$ for all $t \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$, $x \leq x$ if $x \in V$, and $f(\tilde{t}) \leq f(\tilde{s})$ if $t_i \leq s_i$ for all $1 \leq i \leq ar(f)$. Thus $t \leq s$ means 't is less defined than s'. In some way, an Ω-term represents a set of more defined terms $\{s \in \mathcal{T}(\Sigma \cup \{\Omega\}, V) \mid t \leq s\}$. An Ω-term $t = C[\Omega]_u$ in normal form, which represents a term s (i.e., $t \leq s$) having a normal form, has an index u (written $u \in \mathcal{I}(t)$) if it is necessary to further develop the Ω-occurrence u in order to lead s to a normal form. In other words, if $s|_u$ is a redex, it is necessary to reduce it to lead s to a normal form.

Proposition 3.2 ([6]) *Let $t \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$. If $u.v \in \mathcal{I}(t)$, then $u \in \mathcal{I}(t[\Omega]_u)$. If $u \in \mathcal{I}(t)$, $t \leq t'$ and $t'|_u = \Omega$, then $u \in \mathcal{I}(t')$.*

In general, sequential indices are not computable, but Huet and Lévy give a class of TRSs, the strongly sequential TRSs for which this can be done. It is also decidable whether a TRS is strongly sequential. The calculus of a strongly sequential index is performed by using the function ω wich is defined by means of a reduction relation $\to_\Omega$ [6]: Given a TRS $\mathcal{R}$, $C[t] \to_\Omega C[\Omega]$ if $t \neq \Omega$ and $t \leq s$ for some redex s. $\to_\Omega$ is confluent and terminating (see [4, 6]). Then we define $\omega(t)$ to be the $\to_\Omega$-normal form of t. A strongly sequential index of an Ω-term $t \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$ is computed as follows: Assume $u \in O_\Omega(t)$. Let $\bullet$ be a fresh constant symbol. Let $t' = t[\bullet]_u$. Then u is a strongly sequential index of t (written $u \in \mathcal{I}_s(t)$) iff $\bullet$ occurs in $\omega(t')$.

Proposition 3.3 ([6]) *Let $t \in \mathcal{T}(\Sigma \cup \{\Omega\}, V)$ If $u.v \in \mathcal{I}_s(t)$, then $v \in \mathcal{I}_s(t|_u)$.*

Proposition 3.2 also holds for strongly sequential indices by taking $\mathcal{I}_s$ instead of $\mathcal{I}$. By using strongly sequential indices, we define the replacement restriction γ_Ω as follows:

$$\text{for all } t \in \mathcal{T}(\Sigma, V),\ u \in O^{\gamma_\Omega}(t) \text{ iff } u \in \mathcal{I}_s(t[\Omega]_u).$$

We use γ_Ω to motivate the properties introduced to characterize replacement restrictions.

3.2 A set of basic properties

To explore the operational properties of replacement restrictions, we must characterize γ in some way. We give a possible list of properties which can be used to achieve this goal.

ROOT: for all $t \in \mathcal{T}(\Sigma, V)$. $\epsilon \in O^{\gamma}(t)$.
DOWN: for all $C, t \in \mathcal{T}(\Sigma, V)$, $u.v \in O^{\gamma}(C[t]_u) \Rightarrow u \in O^{\gamma}(C[t]_u)$.
SUBTERM: for all $C, t \in \mathcal{T}(\Sigma, V)$, $u.v \in O^{\gamma}(C[t]_u) \Rightarrow v \in O^{\gamma}(t)$.
DECOMP: DECOMP = DOWN + SUBTERM, i.e., for all $C, t \in \mathcal{T}(\Sigma, V)$, $u.v \in O^{\gamma}(C[t]_u)) \Rightarrow (u \in O^{\gamma}(C[t]_u) \wedge v \in O^{\gamma}(t))$.
COMP: for all $C, t \in \mathcal{T}(\Sigma, V)$, $(u \in O^{\gamma}(C[t]_u) \wedge v \in O^{\gamma}(t)) \Rightarrow u.v \in O^{\gamma}(C[t]_u)$.

In the sequel, we motivate our particular selection of properties.

ROOT is immediately satisfied by definition of γ_Ω. Thus $\gamma_\Omega \in \Gamma^{\mathsf{ROOT}}$.

DOWN and SUBTERM borrow Propositions 3.2(1) and Proposition 3.3 respectively. Since DECOMP = DOWN + SUBTERM we have $\gamma_\Omega \in \Gamma^{\mathsf{DECOMP}}$.

If $\gamma \in \Gamma^{\mathsf{DOWN}}$, then $\widetilde{O^{\gamma}}(t)$ is *upward closed* in $(O(t), \leq)$. These properties are useful in implementations: looking for a γ-redex inside a term t, if we found a non-replacing occurrence $u \in \widetilde{O^{\gamma}}(t)$, we can stop the searching for other replacing redexes below.

COMP is related to *transitivity* of indices [11]. A index occurrence v in s is said *transitive* if for any other index occurrence u in t, $u.v$ is an index in $t[s]_u$. Transivity seems to be essential in implementing efficient reduction strategies. It makes feasible to perform the replacement of a redex which is placed on a index occurrence and start the search for a new index locally, i.e., without coming back to the root of the whole term. γ_Ω does not satisfy COMP[1]. If we define γ_τ to be $u \in O^{\gamma_\tau}(t)$ iff u is an occurrence of a transitive index in t, then γ_τ has the property COMP (see Lemma 4.2 in [11]), i.e., $\gamma_\tau \in \Gamma^{\mathsf{COMP}}$. In [11] it has been defined a class of TRSs for which transitive indices are effectively computed. Unfortunately, γ_τ does not inherit the properties of γ_Ω. For instance, DOWN is satisfied by γ_τ, but SUBTERM does not hold, in general.

Another couple of properties which are consequence of the previous ones are:

CONTEXT: for all $C, t \in \mathcal{T}(\Sigma, V)$, $u \in O^{\gamma}(C)$, $v \in O(C)$ such that $u \parallel v$, we get $u \in O^{\gamma}(C[t]_v)$.
REPL: for all $C, t, s \in \mathcal{T}(\Sigma, V)$, $u \in O^{\gamma}(C[t]_u) \Rightarrow u \in O^{\gamma}(C[s]_u)$.
SUBST: for all $t \in \mathcal{T}(\Sigma, V)$, $\sigma \in Subst_{\Sigma}(V)$. $u \in O^{\gamma}(t) \Rightarrow u \in O^{\gamma}(\sigma(t))$. $\widetilde{\mathsf{SUBST}}$: for all $t \in \mathcal{T}(\Sigma, V)$, $\sigma \in Subst_{\Sigma}(V)$. $u \in \widetilde{O^{\gamma}}(t) \Rightarrow u \in \widetilde{O^{\gamma}}(\sigma(t))$.

CONTEXT means that being replacing, for a given occurrence of a term, does not depend on the context *around* it. It stems from Proposition 3.2(2), but CONTEXT is stronger. We have the same for γ_τ (Lema 4.3 in [11]).

REPL means that being replacing in a term t does not depend on subterms *below* the position of the occurrence. By definition of γ_Ω, it easily follows $\gamma_\Omega \in \Gamma^{\mathsf{REPL}}$.

CONTEXT and REPL are important in implementing the restricted rewriting process. They are not independent from the previous properties.

Proposition 3.4 ROOT + COMP + DECOMP $\Rightarrow$ CONTEXT + REPL

SUBST ensures that the application of a substitution does not change the replacing occurrences of the original term. $\widetilde{\mathsf{SUBST}}$ is the analogous for non-replacing occurrences. This is important if we describe the replacement restrictions by starting with some terms, for example, the *lhs*'s of the rules of a TRS and next extending them to instances.

[1] When considering constructor based TRSs, we have a weaker property: for all $C, t \in \mathcal{T}(\Sigma, V)$, such that $t = f(\bar{t})$ and f is a defined symbol, $(u \in O^{\gamma}(C[t]_u) \wedge v \in O^{\gamma}(t)) \Rightarrow u.v \in O^{\gamma}(C[t]_u)$ (see [5, 11]).

Proposition 3.5 CONTEXT + REPL $\Rightarrow$ SUBST. CONTEXT + REPL $\Rightarrow$ $\widetilde{\text{SUBST}}$

Neither γ_Ω nor γ_τ satisfy CONTEXT or SUBST. Thus, sumarizing:

$$\gamma_\Omega \in \Gamma^{\text{ROOT+DECOMP+REPL}}$$
$$\gamma_\tau \in \Gamma^{\text{ROOT+DOWN+COMP+REPL}}$$

The sets Γ^{ROOT}, Γ^{DECOMP}, Γ^{CONTEXT}, and Γ^{REPL} are closed under $\sqcup$ and $\sqcap$. Γ^{COMP} is closed under $\sqcap$. However, as we will see below, Γ^{COMP} is *not* closed under $\sqcup$.

4 Approximating replacement restrictions by context-sensitive restrictions

In order to analyze computational properties of $\hookrightarrow_{\mathcal{R}(\gamma)}$ by only characterizing γ by means of the basic properties of Section 3.2, we approximate Γ by other set of replacement restrictions which have already been analyzed. A trivial approximation of Γ is $\{\gamma_\perp, \gamma_\top\}$: $\hookrightarrow_{\mathcal{R}(\gamma)}$ can always be (upper) approximated by $\rightarrow_{\mathcal{R}} = \hookrightarrow_{\mathcal{R}(\gamma_\top)}$. Some computational properties of $\rightarrow_{\mathcal{R}}$ (for instance, termination) could be 'inherited' by $\hookrightarrow_{\mathcal{R}(\gamma)}$. However, this is not satisfactory, since non-trivial improvements of termination introduced by γ are not captured using this approximation. A more complex approximation domain is needed: the set of context-sensitive restrictions which we define as follows.

A mapping $\mu : \Sigma \rightarrow \mathcal{P}(\mathbb{N})$ is a *replacement map* (or Σ-map) iff for all $f \in \Sigma$. $\mu(f) \subseteq \mathbb{N}^+_{ar(f)}$. $\mu(f)$ determines the *argument* positions which can be reduced for each symbol $f \in \Sigma$ [7]. The ordering $\subseteq$ on $\mathcal{P}(\mathbb{N})$ extends pointwise to an ordering $\sqsubseteq$ on M_Σ, the set of all Σ-maps: $\mu \sqsubseteq \mu'$ if for all $f \in \Sigma$, $\mu(f) \subseteq \mu'(f)$. $(M_\Sigma, \sqsubseteq, \mu_\perp, \mu_\top, \sqcup, \sqcap)$ is a lattice where $\mu_\perp(f) = \emptyset$, $\mu_\top(f) = \mathbb{N}^+_{ar(f)}$, $(\mu \sqcup \mu')(f) = \mu(f) \cup \mu'(f)$ and $(\mu \sqcap \mu')(f) = \mu(f) \cap \mu'(f)$ for all $f \in \Sigma$. $\mu \sqsubseteq \mu'$ means that μ consider less positions than μ' for reduction. Given a Σ-map μ, the predicate $\gamma_{\mu,t}$ defined by $\gamma_{\mu,t}(\epsilon)$ and $\gamma_{\mu,f(t_1,\ldots,t_k)}(i.u) \Leftrightarrow (i \in \mu(f)) \wedge \gamma_{\mu,t_i}(u)$ induces a replacement restriction $\gamma_\mu \in \Gamma$. Γ_{csr} is the set of context-sensitive restrictions $\Gamma_{csr} = \{\gamma_\mu \mid \mu \in M_\Sigma\}$. The lattice $(M_\Sigma, \sqsubseteq, \mu_\top, \mu_\perp, \sqcup, \sqcap)$ induces a lattice $(\Gamma_{csr}, \sqsubseteq, \gamma_{\mu_\top}, \gamma_{\mu_\perp}, \sqcup_{csr}, \sqcap_{csr})$ which is part of $(\Gamma, \sqsubseteq, \gamma_\top, \gamma_\perp, \sqcup, \sqcap)$, but is *not* a sublattice of it: the *lub* operations $\sqcup$ and $\sqcup_{csr}$ differ. We have $\gamma_\mu \sqcup_{csr} \gamma_{\mu'} = \gamma_{\mu \sqcup \mu'}$ and $\gamma_\mu \sqcap_{csr} \gamma_{\mu'} = \gamma_{\mu \sqcap \mu'}$.

Example 4.1 *Let* $t = \mathtt{f}(\mathtt{g}(\mathtt{x},\mathtt{y}), \mathtt{g}(\mathtt{x},\mathtt{y}))$ *be a term. Assume* $\mu(\mathtt{f}) = \mu(\mathtt{g}) = \{1\}$ *and* $\mu'(\mathtt{f}) = \mu'(\mathtt{g}) = \{2\}$. *Then we have* $O^{\gamma_\mu}(t) = \{\epsilon, 1, 1.1\}$ *and* $O^{\gamma_{\mu'}}(t) = \{\epsilon, 2, 2.2\}$. $O^{\gamma_\mu \sqcup \gamma_{\mu'}}(t) = \{\epsilon, 1, 2, 1.1, 2.2\}$, *but* $O^{\gamma_\mu \sqcup_{csr} \gamma_{\mu'}}(t) = O^{\gamma_{\mu \sqcup \mu'}}(t) = \{\epsilon, 1, 2, 1.1, 1.2, 2.1, 2.2\}$.

In general, we have:

Proposition 4.2 *Let* $\gamma_\mu, \gamma_{\mu'} \in \Gamma_{csr}$. *Then* $\gamma_\mu \sqcup \gamma_{\mu'} \sqsubseteq \gamma_\mu \sqcup_{csr} \gamma_{\mu'}$ *and* $\gamma_\mu \sqcap_{csr} \gamma_{\mu'} = \gamma_\mu \sqcap \gamma_{\mu'}$.

Thus, being $\gamma_\mu, \gamma_{\mu'} \in \Gamma_{csr}$ it could be $\gamma_\mu \sqcup \gamma_{\mu'} \notin \Gamma_{csr}$. Note that $\gamma_\top = \gamma_{\mu_\top}$ and $\gamma_\perp \sqsubset \gamma_{\mu_\perp}$ (because $\epsilon \in O^\mu(t)$ for all $\mu \in M_\Sigma$ and term t). In the remainder of this section, we analyze the relationship between context-sensitive restrictions and general restrictions.

We can define (partial) functions $\alpha^{\sqsubseteq}, \alpha^{\sqsupseteq} : \Gamma \rightarrow \Gamma_{csr}$ to give lower and upper context-sensitive approximations of a replacement restriction γ: $\alpha^{\sqsubseteq}(\gamma) \sqsubseteq \gamma \sqsubseteq \alpha^{\sqsupseteq}(\gamma)$. As we will see in Section 5, these approximation functions can be useful to establish computational properties of restricted rewriting by means of the corresponding ones in context-sensitive rewriting. Sometimes, $\alpha^{\sqsubseteq}(\gamma)$ is not defined. This is because, for every μ, and t, $\epsilon \in O^{\gamma_\mu}(t)$.

Proposition 4.3 *$\alpha^{\sqsupseteq}(\gamma)$ exists iff $\gamma \in \Gamma^{\mathsf{ROOT}}$.*

Nevertheless, we can always put an upper bound $\alpha^{\sqsupseteq}(\gamma)$ to a replacement restriction γ, since $\gamma_{\mathsf{T}} = \gamma_{\mu_{\mathsf{T}}}$. This means that $\alpha^{\sqsupseteq}$ can be given as a total function.

We define lower and upper context-sensitive approximations $\gamma_{\mu^{\uparrow}}$ and $\gamma_{\mu^{\downarrow}}$ to a replacement restriction $\gamma \in \Gamma$ as follows: Given a symbol $f \in \Sigma$, (1) If each term $t = f(\tilde{t})$ satisfies $i \in O^{\gamma}(t)$, then we define $i \in \mu^{\downarrow}(f)$. Otherwise, $i \notin \mu^{\downarrow}(f)$. (2) If there is $t = C[f(\tilde{t})]_u$ s.t. $u.i \in O^{\gamma}(t)$, for some i, $1 \leq i \leq ar(f)$, then we define $i \in \mu^{\uparrow}(f)$. Otherwise, $i \notin \mu^{\uparrow}(f)$.

For all $\gamma \in \Gamma$, $\mu^{\downarrow} \sqsubseteq \mu^{\uparrow}$. $\gamma_{\mu^{\downarrow}}$ and $\gamma_{\mu^{\uparrow}}$ bounds γ under some conditions.

Proposition 4.4 *If $\gamma \in \Gamma^{\mathsf{ROOT+COMP}}$, then $\gamma_{\mu^{\downarrow}} \sqsubseteq \gamma$. If $\gamma \in \Gamma^{\mathsf{DOWN}}$, then $\gamma \sqsubseteq \gamma_{\mu^{\uparrow}}$.*

For instance, γ_{τ} can be given a lower bound $\gamma_{\mu^{\downarrow}}$. Both γ_{Ω} and γ_{τ} get upper bounds from the corresponding $\gamma_{\mu^{\uparrow}}$. $\gamma_{\mu^{\downarrow}}$ and $\gamma_{\mu^{\uparrow}}$ are the best *cs*-approximations.

Proposition 4.5 *$\forall \gamma \in \Gamma, \mu \in M_{\Sigma}$, if $\gamma_{\mu} \sqsubseteq \gamma$ then $\gamma_{\mu} \sqsubseteq \gamma_{\mu^{\downarrow}}$. If $\gamma \sqsubseteq \gamma_{\mu}$ then $\gamma_{\mu^{\uparrow}} \sqsubseteq \gamma_{\mu}$.*

Example 4.6 *Let γ be as follows: if $t = \mathtt{h}(\mathtt{g}(\mathtt{f}(\mathtt{a})))$, $O^{\gamma}(t) = \{1.1, 1.1.1\}$ and $O^{\gamma}(s) = \{\epsilon\}$ for any other term $s \neq t$. We have $\mu^{\downarrow}(\mathtt{f}) = \mu^{\downarrow}(\mathtt{g}) = \mu^{\downarrow}(\mathtt{h}) = \emptyset$, and $\mu^{\uparrow}(\mathtt{f}) = \mu^{\uparrow}(\mathtt{g}) = \{1\}$, and $\mu^{\uparrow}(\mathtt{h}) = \emptyset$. Note that $\gamma \notin \Gamma^{\mathsf{DOWN}}$ and $\gamma \notin \Gamma^{\mathsf{COMP}}$. It is easy to see that $\gamma_{\mu^{\downarrow}}$ is* not *a lower bound of γ and also that $\gamma_{\mu^{\uparrow}}$ is* not *an upper bound.*

By using the properties of context-sensitive rewriting as given in [7], and the previous results, we can precisely characterize Γ_{csr} in terms of the properties in Section 3.2.

Theorem 4.7 $\Gamma_{csr} = \Gamma^{\mathsf{ROOT+COMP+DECOMP}}$.

Note that the lack of COMP for γ_{Ω} is essential to put distance between *csr* and γ_{Ω}.

4.1 Decomposition of replacement restrictions

$\gamma_{\mu^{\uparrow}}$ can quickly grow to γ_{T} as long as we have γ-replacing occurrences of arguments in a function symbol f laying in greater and greater contexts. By using the *lub* operation $\sqcup$ of the lattice $(\Gamma, \sqsubseteq, \gamma_{\mathsf{T}}, \gamma_{\perp}, \sqcup, \sqcap)$, there is a different possibility of using context-sensitive replacement restrictions to approximate a replacement restriction γ. If γ is $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_n$, then we use $\sqcup$ to combine upper approximations $\gamma_{\mu_i^{\uparrow}}$ to each component γ_i.

Proposition 4.8 *Let $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_n$ such that and for all j, $1 \leq j \leq n$, $\gamma_j \in \Gamma^{\mathsf{DOWN}}$. Let $\gamma_{\uparrow} = \gamma_{\mu_1^{\uparrow}} \sqcup \cdots \sqcup \gamma_{\mu_n^{\uparrow}}$. Then $\gamma \sqsubseteq \gamma_{\uparrow}$.*

We cannot drop the requirement $\gamma_j \in \Gamma^{\mathsf{DOWN}}$ for components γ_j of $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_n$.

Example 4.9 *Let us consider the term $t = \mathtt{f}(\mathtt{g}(\mathtt{x},\mathtt{y}), \mathtt{g}(\mathtt{x},\mathtt{y}))$. If we split the set $O^{\gamma}(t) = \{\epsilon, 1, 2, 1.1, 2.2\}$ into $O^{\gamma_1}(t) = \{\epsilon, 2, 1.1\}$ and $O^{\gamma_2}(t) = \{\epsilon, 1, 2.2\}$, i.e., $\gamma = \gamma_1 \sqcup \gamma_2$, then we get $\mu_1^{\uparrow}(\mathtt{f}) = \{2\}$, $\mu_1^{\uparrow}(\mathtt{g}) = \{1\}$, and $\mu_2^{\uparrow}(\mathtt{f}) = \{1\}$, $\mu_2^{\uparrow}(\mathtt{g}) = \{2\}$. Then $O^{\gamma_{\mu_1^{\uparrow}}}(t) = \{\epsilon, 2, 2.1\}$ and $O^{\gamma_{\mu_2^{\uparrow}}}(t) = \{\epsilon, 1, 1.2\}$ and $O^{\gamma_{\uparrow}}(t) = \{\epsilon, 1, 2, 1.2, 2.1\}$. Therefore, $\gamma \not\sqsubseteq \gamma_{\uparrow}$.*

Proposition 4.10 *Let $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_n$. Let $\gamma_{\mu_j^{\uparrow}}$, $1 \leq j \leq n$ be the upper context-sensitive approximations for each component γ_j. Then $\gamma_{\mu^{\uparrow}} = \gamma_{\mu_1^{\uparrow}} \sqcup_{csr} \cdots \sqcup_{csr} \gamma_{\mu_n^{\uparrow}}$.*

Proposition 4.10 and Proposition 4.2 amount to saying that $\gamma_\uparrow$ is a better approximation than $\gamma_{\mu\uparrow}$, i.e., $\gamma_\uparrow \sqsubseteq \gamma_{\mu\uparrow}$. In general, $\gamma_\uparrow$ is *not* a context-sensitive restriction. For instance, in Example 4.1, the upper context-sensitive approximation $\gamma_{\mu\uparrow}$ is $\gamma_{\mu\uparrow} = \gamma_{\mu_\top} = \gamma_\top$. If we decompose γ as in Example 4.1, then $\gamma_\uparrow$ is strictly better, since $\gamma_\uparrow \sqsubset \gamma_\top$.

Analogously, whenever we can decompose γ as $\gamma = \gamma_1 \sqcap \cdots \sqcap \gamma_n$, we can also define $\gamma^\downarrow$ to be $\gamma^\downarrow = \gamma_{\mu_1^\downarrow} \sqcap \cdots \sqcap \gamma_{\mu_n^\downarrow}$. We have the corresponding property for $\gamma_{\mu\downarrow}$.

Proposition 4.11 *Let $\gamma = \gamma_1 \sqcap \cdots \sqcap \gamma_n$. Let $\gamma_{\mu_j^\downarrow}$, $1 \leq j \leq n$ be the lower context-sensitive approximations for each component γ_j. Then $\gamma_{\mu\downarrow} = \gamma_{\mu_1^\downarrow} \sqcap_{csr} \cdots \sqcap_{csr} \gamma_{\mu_n^\downarrow}$.*

However, because $\sqcap$ and $\sqcap_{csr}$ coincide in Γ_{csr}, we cannot obtain further refinements by using $\gamma^\downarrow$ instead of $\gamma_{\mu\downarrow}$. In fact, $\gamma^\downarrow = \gamma_{\mu\downarrow}$.

$\gamma_\uparrow$ can be a good approximation to γ. However, the definition of $\gamma_\uparrow$ depend on the particular decomposition of γ as Examples 4.1 and 4.9 show. Fortunately, as we show below, these upper approximations $\gamma_\uparrow, \ldots, \gamma'_\uparrow$ of a replacement restriction γ are closed under $\sqcap$ operation, i.e., $\gamma^\uparrow = \gamma_\uparrow \sqcap \cdots \sqcap \gamma'_\uparrow$ is a (better) upper bound of γ.

4.2 A more accurate approximation domain

We combine replacement restrictions in Γ_{csr} by using the lattice operations to define a new, greater (but still finite) sublattice $\Gamma_{csr}^{\sqcup\sqcap}$ of Γ which can be computed from context-sensitive restrictions. Let $\Gamma_{csr}^{\sqcup\sqcap} = \{\gamma \in \Gamma \mid \exists M_1, \ldots, M_m \subseteq M_\Sigma.\ \gamma = \sqcup_{1 \leq j \leq m} \sqcap_{\mu \in M_j} \gamma_\mu\}$. Being Γ a distributive complete lattice, $\Gamma_{csr}^{\sqcup\sqcap}$ expresses all possible replacement restrictions which can be obtained by combining context-sensitive restrictions. $\Gamma_{csr}^{\sqcup\sqcap}$ admits a simpler formulation. Let $\Gamma_{csr}^{\sqcup}$ be the set $\Gamma_{csr}^{\sqcup} = \{\gamma \in \Gamma \mid \exists \mu_1, \ldots, \mu_n \in M_\Sigma.\ \gamma = \sqcup_{1 \leq j \leq n} \gamma_{\mu_j}\}$. Since $\sqcap = \sqcap_{csr}$, we prove that $\Gamma_{csr}^{\sqcup\sqcap} = \Gamma_{csr}^{\sqcup}$: $\Gamma_{csr}^{\sqcup\sqcap} = \{\gamma \in \Gamma \mid \exists M_1, \ldots, M_m \subseteq M_\Sigma.\ \gamma = \sqcup_{1 \leq j \leq m} \sqcap_{\mu \in M_j} \gamma_\mu\} = \{\gamma \in \Gamma \mid \exists M_1, \ldots, M_m \subseteq M_\Sigma.\ \gamma = \sqcup_{1 \leq j \leq m} \gamma_{\sqcap M_j}\} = \{\gamma \in \Gamma \mid \exists M_1, \ldots, M_m \subseteq M_\Sigma.\ \forall j\ \ 1 \leq j \leq m.\ \mu_j = \sqcap M_j.\ \gamma = \sqcup_{1 \leq j \leq m} \gamma_{\mu_j}\} = \Gamma_{csr}^{\sqcup}$. Therefore, $\sqcap$ is not necessary to run out of Γ_{csr} by using the lattice operations.

Proposition 4.12 *$(\Gamma_{csr}^{\sqcup}, \sqsubseteq, \gamma_{\mu\perp}, \gamma_{\mu\top}, \sqcup, \sqcap)$ is a sublattice of $(\Gamma, \sqsubseteq, \gamma_\perp, \gamma_\top, \sqcup, \sqcap)$. If Γ_{csr} is finite, then $\Gamma_{csr}^{\sqcup}$ is finite.*

$\Gamma_{csr} \subseteq \Gamma_{csr}^{\sqcup}$, but the examples given above show that $\Gamma_{csr}^{\sqcup}$ is a proper superset of Γ_{csr}. In moving from Γ_{csr} to $\Gamma_{csr}^{\sqcup}$, we loose COMP, as the following example shows.

Example 4.13 *Let us consider the term $t = \mathtt{f}(\mathtt{g}(\mathtt{x},\mathtt{y}), \mathtt{g}(\mathtt{x},\mathtt{y}))$. Let us define $\mu(\mathtt{f}) = \mu(\mathtt{g}) = \{1\}$ and $\mu'(\mathtt{f}) = \{2\}$, $\mu'(\mathtt{g}) = \emptyset$. Then we define $\gamma = \gamma_\mu \sqcup \gamma_{\mu'} \in \Gamma_{csr}^{\sqcup}$. Note that $O^\gamma(t) = \{\epsilon, 1, 2, 1.1\}$. Let us consider $t' = t[t]_2$. Then, in spite of the fact that $2 \in O^\gamma(t)$ and $1.1 \in O^\gamma(t)$, $2.1.1 \notin O^\gamma(t') = \{\epsilon, 1, 2, 1.1, 2.2\}$. Thus $\gamma \notin \Gamma^{\mathsf{COMP}}$.*

Thus, COMP is *not* preserved by the $\sqcup$ operation. Using Theorem 4.7 and closedness of $\Gamma^{\mathsf{ROOT+DECOMP+CONTEXT+REPL}}$ under $\sqcup$, it is immediate the following.

Theorem 4.14 $\Gamma_{csr}^{\sqcup} \subseteq \Gamma^{\mathsf{ROOT+DECOMP+CONTEXT+REPL}}$

The following result is an easy consequence of the fact that $\Gamma_{csr}^{\sqcup}$ is a sublattice.

Proposition 4.15 *Let $\gamma \in \Gamma$ and $\gamma_\uparrow, \gamma'_\uparrow \overline{\in} \Gamma_{csr}^{\sqcup}$ such that $\gamma \sqsubseteq \gamma_\uparrow$ and $\gamma \sqsubseteq \gamma'_\uparrow$. Then $\gamma \sqsubseteq \gamma_\uparrow \sqcap \gamma'_\uparrow \in \Gamma_{csr}^{\sqcup}$.*

This gives a method to compute the best upper approximation $\gamma^\uparrow \in \Gamma_{csr}^{\sqcup}$ to a given replacement restriction γ. By doing as many DOWN-decompositions of γ as possible, we obtain a set $\Gamma_\gamma^\uparrow \subseteq \Gamma_{csr}^{\sqcup}$ of upper approximations. Then $\gamma^\uparrow = \sqcap \Gamma_\gamma^\uparrow$ is the best one.

5 Computational properties of restricted rewriting

In this section, we analyze the role of the previously enumerated basic properties in characterizing the computational properties of γ-rewriting by considering the properties of $\hookrightarrow_{\mathcal{R}(\gamma)}$. γ-confluence, γ-termination, etc. of TRSs are just confluence, termination, etc. of $\hookrightarrow_{\mathcal{R}(\gamma)}$. Regarding γ-termination, Proposition 3.1 has a straightforward consequence: if γ, γ' are such that $\gamma \sqsubseteq \gamma'$, then $\mathcal{R}$ γ-terminates if $\mathcal{R}$ γ'-terminates. Since there exist criteria to prove non-trivial termination of context-sensitive rewriting [8, 12], we can use the context-sensitive upper approximation $\alpha^{\sqsupseteq}(\gamma) = \gamma_{\mu\uparrow}$ to prove γ-termination, since $\mathcal{R}$ γ-terminates if it $\alpha^{\sqsupseteq}(\gamma)$-terminates.

COMP implies the preservation of restricted rewriting under replacing contexts, i.e., if $\gamma \in \Gamma^{\mathsf{COMP}}$, $u \in O^{\gamma}(C[t]_u)$ and $t \overset{v}{\hookrightarrow} t'$, then $C[t]_u \overset{u.v}{\hookrightarrow} C[t']_u$. If, additionally, $\gamma \in \Gamma^{\mathsf{REPL}}$, then $t \hookrightarrow^* s$ implies $C[t]_u \hookrightarrow^* C[s]_u$.

If $\gamma \in \Gamma^{\mathsf{SUBST}}$, we get stability of restricted rewriting: $t \hookrightarrow s \Rightarrow \sigma(t) \hookrightarrow \sigma(s)$. Completeness, i.e., the ability of a restriction in computing interesting classes of terms (head-normal forms, values, etc.) is clearly related to the lower approximation $\alpha^{\sqsubseteq}$.

5.1 Confluence of restricted rewriting

In this section, we show how to include replacement restrictions into the theory of rewriting. We obtain a result on γ-confluence by generalizing the *parallel moves lemma*. Given elementary derivations $A : t \overset{u}{\to} t'$ and $B : t \overset{v}{\to} t''$, the parallel moves lemma defines elementary multiderivations[2] $B\backslash A : t' \overset{U}{\to} s$ and $A\backslash B : t'' \overset{V}{\to} s$ which *converge* to a common reduct s [4]. An immediate consequence is confluence of orthogonal TRSs.

To generalize this result, we analyze the requirements of the parallel moves lemma. The starting point is the concept of *residual* of a γ-redex. Given a TRS $\mathcal{R} = (\Sigma, R)$ and an elementary derivation $A : t \overset{[u,\alpha]}{\to} s$, where $\alpha : l \to r \in R$, and $v \in O_{\mathcal{R}}(t)$, the set $v\backslash A$ of *residuals* of redex $t|_v$ by A is a subset of $O(s)$ as follows [4]:

$$v\backslash A = \begin{cases} \emptyset & \text{if } v = u \\ \{v\} & \text{if } v \parallel u \qquad (i) \\ \{v\} & \text{if } v < u \qquad (ii) \\ \{u.w_1.v_1 \mid r|_{w_1} = x\} & \text{if } v = u.w.v_1 \text{ and } l|_w = x \in V \quad (iii) \end{cases}$$

For any nonelementary derivation A, we define $v\backslash A$ to be $v\backslash 0 = \{v\}$ (0 is the empty derivation), and $v\backslash(AB) = \{w\backslash B \mid w \in v\backslash A\}$. $u\backslash A$ is extended to sets of redex occurrences as follows: $U\backslash A = \bigcup_{u \in U} u\backslash A$. Given derivations A, B, starting from t and being B an elementary derivation contracting the set $U \subseteq O_{\mathcal{R}}(t)$, the *residual* derivation $B\backslash A$ of B by A is the elementary derivation contracting the set $U\backslash A$.

The fact $v\backslash A \subseteq O_{\mathcal{R}}(s)$ is essential to define the concept of residual derivation, etc. It means that *the derivation A (which contracts the redex occurrence u of t) left unchanged the possibility of reducing the (descendants of the) redex v in further reduction steps.*

We need to keep the analogous property in γ-restricted derivations A^{γ}: $v \in O^{\gamma}_{\mathcal{R}}(t) \Rightarrow v\backslash A^{\gamma} \subseteq O^{\gamma}_{\mathcal{R}}(s)$. We refer to this property as P_{γ}. We give the following definition.

Definition 5.1 *Given an elementary γ-derivation $A^{\gamma} : t \overset{[u,\alpha]}{\hookrightarrow}_{\mathcal{R}(\gamma)} s$, where $\alpha : l \to r \in R$, and $v \in O^{\gamma}_{\mathcal{R}}(t)$, the set $v\backslash A^{\gamma}$ of* residuals *of v by A^{γ} is a subset of $O(s)$ as follows: $v\backslash A^{\gamma} = v\backslash A$.*

[2] An elementary multiderivation, simultaneously contracts a set U of disjoint occurrences of redexes. We write $t \Vdash s$ when the information about the concrete contracted occurrences is not relevant.

Since any γ-derivation is also a derivation, the previous definition is correct. However a residual of a replacing redex does not need to be a replacing redex.

Example 5.2 *Let us consider the following orthogonal TRS $\mathcal{R}$.*

$\mathtt{f(x)} \to \mathtt{g(x,x)}$

$\mathtt{h(0)} \to \mathtt{0}$

If we define $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{g}) = \{1\}$, then we have the following γ_μ-derivation:

$A^{\gamma_\mu} : \underline{\mathtt{f(h(0))}} \hookrightarrow \mathtt{g(h(0),h(0))}$

The residuals of the replacing redex $\mathtt{h(0)}$ in $t = \mathtt{f(h(0))}$ are $\{1,2\}$. However, $2 \notin O^{\gamma_\mu}(\mathtt{g(h(0),h(0))})$ and it is not *a replacing redex.*

The requirement of having left homogeneous γ-replacing variables solves the problem.

Definition 5.3 (TRS with left homogeneous replacing variables) *Let $\mathcal{R} = (\Sigma, R)$ be a TRS and $\gamma \in \Gamma^{\mathsf{COMP+SUBST}}$ be a replacement restriction. A rule $\alpha : l \to r \in R$ has* left *homogeneous γ-replacing variables, if, for all $x \in Var^\gamma(l)$, $O_x(l) = O_x^\gamma(l)$ and $O_x(r) = O_x^\gamma(r)$. $\mathcal{R}$ has left homogeneous γ-replacing variables, if all rules in R have left homogeneous γ-replacing variables.*

For left-linear TRSs, the requirement of *left* homogeneous γ-replacing variables is simpler: for all $x \in Var^\gamma(l)$, $O_x(r) = O_x^\gamma(r)$. TRSs having homogeneous γ-replacing variables are expected to hold the following: instances of replacing variables in *lhs*'s are replacing in *rhs*'s. $\gamma \in \Gamma^{\mathsf{SUBST}}$ ensures this property. COMP ensures the property in any context.

When considering TRSs having left homogeneous replacing variables, residuals of replacing redexes are also replacing. Then the extensions of the notion of *residual* are sound. In particular the notion of residual derivation $B \backslash A$. Indeed, if we cannot ensure $v \backslash A^\gamma \subseteq O_{\mathcal{R}}^\gamma(s)$, then, we cannot ensure that $U \backslash A^\gamma \subseteq O_{\mathcal{R}}^\gamma(s)$. Since the residual derivation $B^\gamma \backslash A^\gamma$ is a derivation contracting the set $U \backslash A^\gamma$, this would not be well defined, because this set could contain non replacing redexes.

In orthogonal TRSs, the contraction of a redex occurrence does *not* eliminate existing redexes. This is because of *lhs*'s of rules do not overlap. Regarding γ-redexes, we can also ensure the desired property by forbidding the replacement of overlapping redexes.

Definition 5.4 (γ-overlapping terms) *Let $t, s \in \mathcal{T}(\Sigma, V)$ and $\gamma \in \Gamma$. s γ-overlaps t at the occurrence u, if $u \in O_\Sigma^\gamma(t)$ and $t|_u$ and s unify. Terms t, s are* not *γ-overlapping, if neither s γ-overlaps t nor t γ-overlaps s*

The generalization of orthogonal TRSs in presence of a replacement restriction γ is:

Definition 5.5 (γ-orthogonal TRS) *Let $\gamma \in \Gamma^{\mathsf{SUBTERM+\widetilde{SUBST}}}$. Let $\mathcal{R}$ be a left-linear TRS. $\mathcal{R}$ is γ-orthogonal if there is no γ-overlapping lhs's.*

$\gamma \in \Gamma^{\mathsf{SUBTERM+\widetilde{SUBST}}}$ ensures that non γ-replacing occurrences of *lhs*'s are not γ-replacing in instances of the *lhs*. $\gamma \in \Gamma^{\mathsf{SUBTERM}}$ ensures that this fact is keeped when the redex occurrences are in any context C. If we would have a term $t = C[\sigma(l)]_u$ having an overlapping γ-redex at an occurrence $u.v \in O^\gamma(t)$, with $v \in \widetilde{O^\gamma}(l)$, then we would get, by $\mathsf{SUBTERM}$ $v \in O^\gamma(\sigma(l))$. But $\widetilde{\mathsf{SUBST}}$ ensures that $v \notin O^\gamma(\sigma(l))$ and we get a contradiction. Now we characterize a class of replacement restrictions which satisfy P_γ.

1. From (i), we require CONTEXT. This is because, after the replacement performed by A to the occurrence $u \in O^{\gamma}_{\mathcal{R}}(t)$, $u \parallel v$, it must be (in order to satisfy P_{γ}), $v \backslash A^{\gamma} = \{v\} \subseteq O^{\gamma}_{\mathcal{R}}(s)$. If $\gamma \in \Gamma^{\mathsf{CONTEXT}}$, then this is true.

2. From (ii), we require REPL. In this way, the replacement performed by A which occurs *below* v, since $v < u$, keeps $s|_v$ being a redex (by orthogonality) and, moreover a γ-redex (by REPL).

3. From (iii), we require that $\mathcal{R}$ has left homogeneous γ-replacing variables and also $\gamma \in \Gamma^{\mathsf{SUBTERM+COMP+REPL}}$. SUBTERM ensures that v_1 is a replacing occurrence of $t|_{u.w}$. If $x \in Var^{\gamma}(l)$ and $\mathcal{R}$ has left homogeneous γ-replacing variables, then $w_1 \in O^{\gamma}(s|_u)$. REPL ensures that u is replacing after the replacement. Finally, COMP ensures P_{γ}.

4. Orthogonality is also required to justify (iii). Otherwise, we are not sure to express v as $v = u.w.v_1$ where $l|_w$ is a variable. But we can relax this requirement to γ-orthogonality.

Thus, we have shown the following (note that, by Proposition 3.5, since we impose CONTEXT and REPL, we do not need to further require SUBST and $\widetilde{\mathsf{SUBST}}$).

Proposition 5.6 *Let $\gamma \in \Gamma^{\mathsf{SUBTERM+COMP+CONTEXT+REPL}}$ be such that $\mathcal{R} = (\Sigma, R)$ is γ-orthogonal and has left homogeneous γ-replacing variables. Let $A^{\gamma} : t \overset{[u,\alpha]}{\hookrightarrow}_{\mathcal{R}(\gamma)} s$ be an elementary γ-derivation, where $\alpha : l \to r \in R$, and $v \in O^{\gamma}_{\mathcal{R}}(t)$. Then $v \backslash A^{\gamma} \subseteq O^{\gamma}_{\mathcal{R}}(s)$.*

Now we give the generalization of the parallel moves lemma for restricted rewriting.

Lemma 5.7 *Let $\mathcal{R}$ be a TRS and $\gamma \in \Gamma^{\mathsf{SUBTERM+COMP+CONTEXT+REPL}}$, such that $\mathcal{R}$ is γ-orthogonal and has left homogeneous γ-replacing variables. Let A^{γ}, B^{γ} be elementary gamma-multiderivations starting from a term t. Then $B^{\gamma}(A^{\gamma} \backslash B^{\gamma})$ and $A^{\gamma}(B^{\gamma} \backslash A^{\gamma})$ are derivations starting from t and leading to the same term s and, for all $u \in O^{\gamma}_{\mathcal{R}}(t)$, we have $u \backslash B^{\gamma}(A^{\gamma} \backslash B^{\gamma}) = u \backslash A^{\gamma}(B^{\gamma} \backslash A^{\gamma})$.*

A relation $R \subseteq A \times A$ is said strongly confluent if, for all $a, b, c \in A$, $a \; R \; b \wedge a \; R \; c \Rightarrow b \; R^{*} d \wedge c \; R^{\epsilon} \; d$ for some $d \in A$, where R^{ϵ} is the reflexive closure of R, $R^{\epsilon} = R \cup \{(a, a) \mid a \in A\}$. Lemma 5.7 proves that $\Vdash\!\!\!\rightarrow$ is strongly confluent. In order to conclude confluence of $\hookrightarrow$, we use the fact $\hookrightarrow^{+} = \Vdash\!\!\!\rightarrow^{+}$. This can be proved if we impose $\gamma \in \Gamma^{\mathsf{CONTEXT}}$. Now we can use the following fact [3]: Let $R, S \subseteq A \times A$ be relations such that $R^{+} = S^{+}$. If S is strongly confluent, then R is confluent. We get the desired result.

Theorem 5.8 *Let $\mathcal{R}$ be a TRS and $\gamma \in \Gamma^{\mathsf{SUBTERM+COMP+CONTEXT+REPL}}$, such that $\mathcal{R}$ is γ-orthogonal and has left homogeneous γ-replacing variables. Then $\mathcal{R}$ is γ-confluent.*

As an immediate application of Theorems 5.8 and 4.7, by taking into account Proposition 3.4, we obtain a new result on γ_{μ}-confluence of *csr* under a replacement map μ, which complements the results in [7] which only applies to γ_{μ}-terminating programs.

Theorem 5.9 *Let $\mathcal{R}$ be a TRS and μ be a replacement map such that, such that $\mathcal{R}$ is γ_{μ}-orthogonal and has left homogeneous γ_{μ}-replacing variables. Then $\mathcal{R}$ is γ_{μ}-confluent.*

6 Conclusions

We have given a framework to analyze arbitrary replacement restrictions by characterizing them by means of a set of properties. We have shown that the class of context-sensitive restrictions can be precisely described by using these properties and we show how to give the best lower and upper context-sensitive approximations to arbitrary replacement restrictions. This is useful to test computational properties of restricted rewriting by comparison with *csr*, whose computational properties have already been analyzed. Conversely, we can also obtain new results for *csr*. We have also exemplified how to generalize results on computational properties of rewriting to restricted rewriting by using the basic properties which characterize the latter. Our framework is useful to obtain a rapid characterization of computational properties of a reduction relation induced by a given replacement restriction. Since our results do not depend on the particular definition of a replacement restriction, if we characterize it by means of the properties of Section 3.2, we can immediately use the results of this paper to obtain information about its properties.

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